A canonical Feynman path integral quantization of the Schwarzschild metric

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Abstract

The Schwarzschild metric may be quantized under the usual assumptions of path integral quantization as well as general relativity, if the black holes are treated as single quantum objects representing composite systems of gravitons. We assume that a black hole emits a system of massless gravtions, which therefore follow the same frequency to energy relation as the photon. We assume conservation of energy-momentum and enforce it "by hand" in the act of gravitational radiation. The rest mass of the black hole imposes a high energy cutoff scale. Approximate real quantum effects outgoing from the event horizon are nonzero and normalizable. The quantum gravitional field oscillators satisfy the boundary conditions of the Einstein-Hilbert action over outgoing null intervals with a past boundary on the event horizon.

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1 INTRODUCTION

Applying the canonical quantum theory to the canonical modern theory of gravity exactly, with no or few new axioms, is regarded as extremely difficult. Quantizing the hypothetical graviton itself leads to nonrenormalizable infinities that preclude a computationally useful physical description of gravity. "Canonical quantum gravity" proper follows the Hamiltonian formulation of mechanics. It was established decades ago [1][5], and Hamiltonian approaches based on this early work are still being actively researched up to today [12]. The Feynman path integral formulation of quantum mechanics applied to general relativity has led to useful results, as well [7]. Despite the difficulties with quantizing the graviton directly, a minisuperspace approximation scheme can also at least give a normalizable canonical result, such as in the context of the quantum geometry of the Schwarzschild solution [2].

The Feynman path integral formulation of quantum mechanics is a Hamilton-Jacobi approach that allows us to quantize by assigning phases to mechanical paths by their classical actions. "Naively," one expects a canonical quantum gravity treatment, equivalent to the Hamiltonian approach, to follow from applying path integral quantization to the Einstein-Hilbert action of general relativity. This is fraught with problems. The accepted quantum and relativistic theories might even be intrinsically ambiguous on points of how to do so. However, the quantization of a black hole as a single composite object with many gravitons leads to a useful description.

Quasi-independent gravitational quantum variation is allowed to occur within regions that satisify the boundary conditions on the past boundary of the event horizon. These past-bounded harmonics can be treated similarly to the minisuperspace approximation. To achieve a normalizable description, conservation of energy and momentum must be inserted into the path integral "by hand." Satisfying these boundary conditions and enforcing conservation principles, a numerically useful canonical approach results, with no new major assumptions.

We will attempt to form a composite path integral description of the Schwarzschild solution, with spatially localized gravitational radiation, under the condition that the apparent Schwarzschild radius might vary locally, quantum mechanically.

(Note that we take Planck units as natural in all that follows.)

2 Methods

Our goal is to locally quantize the geometry of the Schwarzschild solution, so we allow the Schwarzschild radius r_s to vary locally from region to region. We call the varying parameter "$\mathscr{r}_s$." We understand this as a scale parameter that describes the local appearance of the geometry in terms of the classical solution as a reference. We first define $\mathscr{r}_s$ as the sum of the classical radius r_s and a perturbation $\mathscr{r}_c$.

Consider the variation of the Einstein-Hilbert action:

$$\delta A_{EH} = \int_V d^4x\sqrt{-g}\left(G_{\mu\nu}\delta g^{\mu\nu}\right) + \int_V d^4x\sqrt{-g}\nabla_\epsilon\left(g^{\mu\nu}\delta\Gamma^\epsilon{}_{\mu\nu} - g^{\epsilon\eta}\delta\Gamma^\mu{}_{\mu\eta}\right) \quad (1)$$

where $G_{\mu\nu}$ is the inverse Einstein tensor, $\Gamma^\epsilon{}_{\mu\nu}$ are the Christoffel symbols, $g^{\mu\nu}$ is the inverse of the metric tensor, and g is the metric tensor trace [11]. It is common to assume that, due to Stokes' theorem, the second integral is a boundary term, which we can take to vanish at infinity [4]. However, over a finite volume, we must include the value at its boundary. Some research suggests this boundary and its interior bulk are holographically dual [10].

Following from equation 1, in order for the Einstein-Hilbert action to be independent over a region, the boundary term must vanish. Rewritten from equation 1, this term is

$$\delta A_{boundary} = \int_v d^3x\sqrt{h}\left(Kh^{ij} - K^{ij}\right)\delta h^{ij} \quad (2)$$

where h^{ij} is the inverse induced 3-space metric, h is its trace, and K^{ij} is the inverse extrinsic curvature [11].

It can be shown with rigor that no gravitational effect in general relativity can propagate faster than the speed of light [3], so we assume that our gravitational quantum variations should propagate along null geodesics. We therefore expect quantum variations from the classical geometry, $\mathscr{r}_c$, to take the form of outgoing waves travelling at the speed of light. We define t' as a retarded time coordinate:

$$t' \simeq \left(t - r + r_s(t,r) + r_s(t,r)\ln\left(\frac{r}{r_s(t,r)}\right)\right). \quad (3)$$

We have ignored the wave contribution from our ultimate $\mathscr{r}_c$, which should average to zero if we satisfy the boundary conditions of the action. We have chosen the constant of integration such that $t' = t$ on the event horizon of the black hole, so that the coordinate parameterizes outgoing null geodesics from the event horizon. (We restrict our consideration here to the exterior of the black hole.) r_s can be made an arbitrary function solely of t', since changes in the apparent Schwarzschild radius spread at the speed of light, but the recursive definition of t' can be problematic.

Massless gravitons should have wavelength $\sqrt{1 - \frac{r_s(t,r)}{r}}kt'$, with wavenumber k. (The additional factor in front keeps proportionality with the proper time interval passing when holding r fixed.) We are quantizing a system of many gravitons, so we expect their amplitude functions to carry $\hbar k$ units of energy per particle. (Only discrete particles should be emitted.) If we think of these gravitons as apparent oscillations of the rest mass of the black hole, an oscillation between twice its mass and 0 should carry the entire rest mass energy of the hole. The amplitudes should be constant on radially symmetric wavefronts of constant t', and this also conserves momentum:

$$\mathscr{r}_q = \mathscr{r}_c + r_s(t') \simeq \int_0^\infty 2kb(k,t') \sin\left(\sqrt{1 - \frac{r_s(t,r)}{r}}kt'\right) dk + r_s(t'). \tag{4}$$

where the factor of 2 relates rest mass to r_s. This variation implies $\frac{dr}{dt} = \left(1 - \frac{r_s(t')}{r}\right)$ for the waves at any point along their paths, which is the radial coordinate speed of light in Schwarzschild coordinates. We have ignored a recursive contribution of $\mathscr{r}_c$ itself to the sinosoidal argument because it averages to zero over full wavelengths, so its effect on the line element can be ignored if we may choose limits of integration containing only full wavelengths.

Satisfying this boundary constraint, we should be able to quantize via a Feynman path integral, parameterized by the amplitude density $b(k,t')$. For any volume V that satisfies the action's boundary constraints, the Einstein-Hilbert action between time coordinates t_n and t_m is

$$S(n,m) = \frac{1}{16\pi G_N} \int_{t_n}^{t_m} \int_V R\sqrt{-g}\, d^3x\, dt \tag{5}$$

[6], where R is the Ricci scalar, $\sqrt{-g}$ is the volume element, and G_N is Newton's constant. Neglecting normalization, the quantized form is

$$\int d\vec{\mathscr{r}}_{sN-1} \int d\vec{\mathscr{r}}_{sN-2} \ldots \int d\vec{\mathscr{r}}_{s2} \prod_{n=2}^{N} \exp\left(\frac{iS(n,n-1)}{\hbar}\right) \tag{6}$$

[13], where we have replaced the usual integrals over position with integrals over a configuration "vector" for the value of $\mathscr{r}_s$ over all space at a time step.

We have explicitly formed the Einstein-Hilbert action, the surface term, and a Feynman path integral of the action, and analyzed them symbolically and numerically with Wolfram Mathematica and a differential geometry package [8]. One aim was to verify their agreement with the established theory via the Euler-Lagrange equation by recovering the classical gravitational system when the functional derivative of the action vanishes. We also determined what other conditions need to be fixed initially for the classical solution to result from the Euler-Lagrange equation. Ultimately, we sought a normalizable Feynman path integral expression to be used in numerical simulation of quantum gravity. The spectrum of allowable states can be normalized if we assume that the black hole cannot emit more gravitational radiation than its rest mass and that a de Broglie wavelength (for a presumed massless graviton) can be inserted by hand into the system to discretize the allowable amplitudes.

3 Results

For the Schwarzschild metric, allow $\mathscr{r}_c$ to be arbitrarily parameterized over t, r, θ, and ϕ. In order to ultimately enforce conservation of energy, allow r_s to be arbitrarily parameterized over t. In this case, the integrand of the boundary term is diagonal. It must vanish on the surface of the region to quantize.

It was checked with a Wolfram Mathematica differental geometry package [8] that the boundary term vanishes for this pertubation on all surfaces of constant r and on any arbitrary pair of constant θ and contant ϕ faces with equal circumference fraction. We recover agreement with the classical Euler-Lagrange equations at least over full wavelengths of the perturbation along r, so that we may ignore recursive wave contributations. For a totally general quantum variation, we must take the action integral over the full interior or exterior domain of r of the black hole for the functional derivative of the classical path to be 0 as we expect.

The resulting Einstein-Hilbert action is straightforward to form with symbolic mathematical analysis software:

$$\frac{1}{16\pi}\int_0^{\pi}\int_0^{2\pi} R\sqrt{-g}\,d\theta\,d\phi = \frac{1}{4(r-\mathscr{r}_q)^2}\frac{\partial^2 r_s}{\partial t^2} + \frac{r^3}{2(r-\mathscr{r}_q)^3}\left(\frac{\partial r_s}{\partial t}\right)^2 + \frac{r}{4}\frac{\partial^2 r_s}{\partial r^2} + \frac{1}{2}\frac{\partial r_s}{\partial r} \tag{7}$$

This expression seems tractable, but to simplify further, remember that $\mathscr{r}_q$ can be replaced by r_s under appropriate limits of integration. Defining an advanced radial coordinate r' analogous to the retarded time t', we can transform the partial derivatives. (We give the action in terms of t' and r here after transforming the partial derivatives, which is simpler:)

$$\frac{1}{16\pi}\int_0^{\pi}\int_0^{2\pi} R\sqrt{-g}\,d\theta\,d\phi = \frac{r^3}{8(r-r_s(t'))^3}\left(\frac{dr_s}{dt'}\right)^2 + \frac{(r-r_s(t'))}{4r}\frac{dr_s}{dt'} + \frac{\left(2r^4 - 4r^3 r_s(t') + 6r^2 r_s(t')^2 - 4r r_s(t')^3 + r_s(t')^4\right)}{16r(r-r_s(t'))^2}\frac{d^2 r_s}{dt'^2} \tag{8}$$

Notice that the action is vacuum for vanishing r_s time derivatives. Although the action is zero in that case, remember that each variation is a distinct (vacuum) metric. When r_s does not change, any configuration of gravitational waves is an extremum of the action that is energy degenerate with the classical Schwarzschild solution. (Recall that the Einstein field equations do not directly track the gravitational contribution to the stress-energy-momentum.) Also remember that that $r_s(t')$ is arbitrary in a Feynman path integral, and that it implies information about r_s traveling at exactly the speed of light.

Our generalized position appears to be the collective graviton amplitude, and our generalized momentum appears to be proportional to the change in the collective amplitude, which implies a change in r_s. This suggests noncommuting amplitude and amplitude time derivative operators in the Hamiltonian formulation.

When the derivatives of r_s vanish, so does the action. This implies that any and all configurations of the vacuum field are solutions to the Einstein field equations, with degenerate action equal to the classical Schwarzschild solution's. However, "keeping book by hand," the black hole's Schwarzschild radius should have changed in the past due to radiation, tracing a radiative surface back in time to $t' = t$. As such, the past mechanics are complicated and imply real radiated gravitons. If the Schwarzschild radius cannot change, though, as we might expect for fundamental quantum particles,

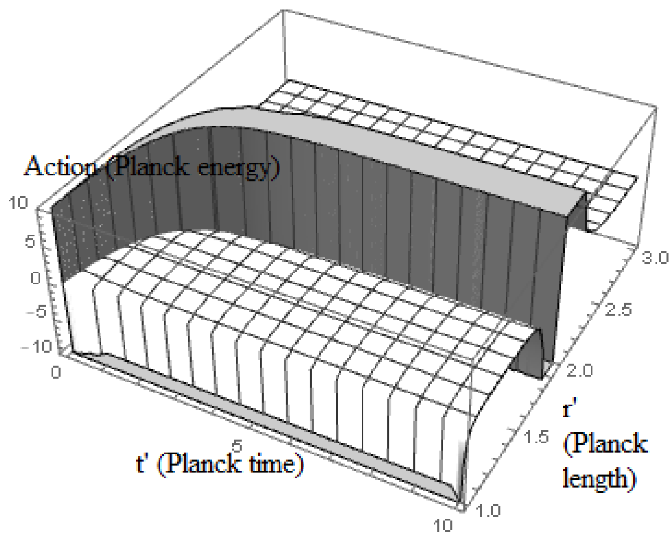

Figure 1: *Schwarzschild action vs. retarded time and advanced radius for* $r_s = 1$, $\frac{dr_s}{dt'} = 0.5$, *and* $\frac{d^2 r_s}{dt'^2} = 0.5$, *(with* $c = 1$, $G_N = 1$, $\hbar = 1$*)*

gravitons must be immediately radiated back out when absorbed. In the case of constant r_s, define an incoming advanced time coordinate:

$$t'' \simeq \left(t + r - r_s - r_s \ln\left(\frac{r}{r_s}\right)\right), \tag{9}$$

where, again, $t'' = t$ on the event horizon. Add an incoming r_s perturbation to our original $\mathscr{r}_q$:

$$\mathscr{r}_q = \mathscr{r}_c + r_s \simeq r_s + \int_0^\infty 2kb(k,t') \sin\left(\sqrt{1 - \frac{r_s}{r}} kt'\right) dk + \int_0^\infty 2kb(k,t'') \sin\left(\sqrt{1 - \frac{r_s}{r}} kt''\right) dk. \tag{10}$$

If the gravitating object cannot change its rest mass, as is ostensibly the case with a fundamental quantum particle, then incoming radiation must be immediately turned out as outgoing. Our incoming radiation is unrealistically symmetric, but remember that we have constrained the net momentum (and angular momentum) as zero, requiring this symmetry. This form of the perturbation preserves time reversal symmetry and therefore represents an infinite family of solutions to the Einstein field equations with vacuum action equal to the classical solution's.

Common definitions of the gravitational energy, such as via the Landau-Lifshitz pseudotensor [9], make it equal to $-r_s/2$ in total in this case, such that the gravitons should not be able to radiate more negative energy than this. Gravitons should have their stress-energy-momentum counted in one or more appropriately chosen pseudotensor contributions, and a given pseudotensor should maintain consistence with the distribution of tensorial stress-energy-momentum. This picture is consistent with a gravitational wave carrying negative energy in the pseudotensor contribution while imparting positive kinetic energy to a test oscillator with rest mass. Therefore, any real gravitational radiation via our perturbation should only occur with radiation of (positive) gravitating rest mass. This implies a reversal of the overall sign of the simple harmonic oscillator action, which has no effect on the (extremal) classical equations of motion. An additional constraint is implied between the amplitude functions and $r_s(t')$, that the change in the gravitational energy cannot cause the black hole to radiate more than its total classical energy. The amplitudes $b(k, t')$ imply an energy for the gravitational radition. Further, the speed of light propagation of b implies a form $b(k, t, r)$ The energy radiated can be found from a surface integral of the amplitude flux at the event horizon. The integral is trivial, since the number functions already imply particles spread over spherical shells with volume:

$$\frac{dE}{dt} = \int_0^\infty \hbar k \frac{\partial b(k,t,r)}{\partial t}\bigg|_{r=r_s(t,r)} dk. \tag{11}$$

(The only surface of interest in this case is the event horizon) This constrains the parameterization of r_s:

$$\frac{dr_s}{dt} = -2G_N \frac{dE}{dt}. \tag{12}$$

Addtionally, the amplitude functions are further constrained by the requirement that $r_s(t')$ is never less than zero, i.e. that the black hole can't emit more than its total rest mass energy equivalent in the act of gravitational radiation. Careful numerical inspection reveals that this does not equal the energy implied by the Einstein-Hilbert action, but remember that the Einstein field equations track a stress-energy-momentum tensor without gravitational contribution and that energy is not locally conserved without accounting for this contribution, such as by an appropriate pseudotensor.

4 DISCUSSION

Computational verification of the vanishing boundary term is straightforward and was accomplished with a direct insertion of the ansatz into the boundary constraints with a Mathematica differential geometry package [8]. The significance of a vanishing boundary term is that the bounded region may be varied smoothly but independently while holding the boundary fixed, implying infinitesimal field oscillators.

We can quantize the radiation, i.e. the action with vanishing r_s time derivatives, with a Feynman path integral of the Einstein-Hilbert action. For the case of vanishing derivatives, such as when the rest mass of a fundamental quantum particles is held fixed, the result is trivial: all values of the perturbation give the same action as the classical solution. As such, all field configurations should be equally likely around a gravitating object that cannot radiate mass, such as a stationary proton as in the hydrogen model. In the case of real radiation, we enter the action given above into a path integral and vary $r_s(t')$ arbitrarily under the constraint that information about changes in the radius propagate at the speed of light. The high energy end of the theory is avoided since a maximum k is imposed by the requirement that r_s cannot be less than 0.

The quantum objects of our treatment are regions of spacetime rather than discrete gravitons. That is, our approach has been to quantize a macroscopic system with many gravity particles without direct recourse to the fundamental gravitational field. The use of a quantum formalism describing a boson is appropriate, for the statistics seem to be immaterial to the distinguishable quantized spacetime regions, regardless of whether they would be appropriate for the graviton itself. For a massless graviton, which travels like light, a reasonable form for the energy per quantum appears to be the usual $E = \hbar k$ of a massless quantum particle. However, as per usual for the Einstein field equations, local conservation is not apparent without a pseudotensor contribution to the stress-energy-momentum.

5 CONCLUSIONS

We can treat gravitating systems quantum mechanically if we assume a composite quantum object with internal degrees of freedom for a discrete, massless, bosonic graviton gas. We also must perform certain "book-keeping" of conserved quantities in an arguably ad hoc manner, though the result therefore properly conserves. It is appropriate to ignore wave-like recursive quantum contributions of a time-varying Schwarzschild radius over limits of integration which recover on average the classical Euler-Lagrange equation for general relativity, i.e. over full variation wavelengths. The expression for the action in the implied Feynman path integral shares the essential transformation invariances of the classical system, since it is written in terms of the classical coordinates rather than operators. (However, all modes of gravitational radiation need to be treated to show full exact Lorentz invariance.) The perturbed metric with matched incoming and outgoing waves represents an infinite family of solutions to the Einstein field equations with energy degenerate to the classical solution. Momentum, angular momentum and possibly stress quantum variations of black holes must still be treated fully in a complete theory of gravitationally interacting point-like particle mechanics.

References

[1] Arnowitt, R., Deser, S., and Misner, C. W. Republication of: The dynamics of general relativity. *General Relativity and Gravitation*, 40 (9):1997–2027, 2008.

[2] Ashtekar, A. and Bojowald, M. Quantum geometry and the Schwarzschild singularity. *Classical and Quantum Gravity*, 23(2):391, 2005.

[3] Carlip, S. Aberration and the speed of gravity. *Physics Letters A*, 267 (2):81–87, 2000.

[4] Carroll, S. M. *Spacetime and geometry. An introduction to general relativity*, volume 1. 2004. p.193.

[5] Dirac, P. A. The theory of gravitation in Hamiltonian form. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 246, pages 333–343. The Royal Society, 1958.

[6] Feynman, R. P. *Feynman Lectures on Gravitation.* Addison-Wesley Publishing, 1995.

[7] Hawking, S. W. The path-integral approach to quantum gravity. In *General relativity*. 1979.

[8] Headrick, M. A Mathematica package for tensor algebra and calculus. http://goo.gl/UDso5q, 2013. Accessed: 2015-12-16.

[9] Landau, L. and Lifshitz, E. *The Classical Theory of Fields*. 1951.

[10] Padmanabhan, T. Holographic gravity and the surface term in the Einstein-Hilbert action. *Brazilian Journal of Physics*, 35(2A):362–372, 2005.

[11] Padmanabhan, T. A short note on the boundary term for the Hilbert action. *Modern Physics Letters A*, 29(08):1450037, 2014.

[12] Rovelli, C. Loop quantum gravity. *Living Rev. Rel*, 1(1):41–135, 1998.

[13] Sakurai, J. J. and Napolitano, J. *Modern Quantum Mechanics*. Second edition.